

GROTHENDIECK–LEFSCHETZ TYPE THEOREMS FOR THE LOCAL PICARD GROUP

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A special case of the Lefschetz hyperplane theorem asserts that if X is a smooth projective variety and $H \subset X$ an ample divisor then the restriction map $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(H)$ is an isomorphism for $\dim X \geq 4$ and an injection for $\dim X \geq 3$.

If X is normal, then the isomorphism part usually fails. Injectivity is proved in [Kle66, p.305] and an optimal variant for the class group is established in [RS06].

For the local versions of these theorems, studied in [Gro68], the projective variety is replaced by the germ of a singularity ($p \in X$) and the ample divisor by a Cartier divisor $p \in X_0 \subset X$. The usual (global) Picard group is replaced by the *local Picard group* $\mathrm{Pic}^{\mathrm{loc}}(p \in X)$; see Definition 7.

Grothendieck proves in [Gro68, XI.3.16] that if $\mathrm{depth}_p \mathcal{O}_X \geq 4$ then the map between the local Picard groups $\mathrm{Pic}^{\mathrm{loc}}(p \in X) \rightarrow \mathrm{Pic}^{\mathrm{loc}}(p \in X_0)$ is an injection. Note that this does not imply the Lefschetz version since a cone over a smooth projective variety usually has only depth 2 at the vertex.

The aim of this note is to propose a strengthening of Grothendieck’s theorem that generalizes Kleiman’s variant of the global Lefschetz theorem. Then we prove some special cases that have interesting applications to moduli problems.

Problem 1. Let X be a normal (or S_2 and pure dimensional) scheme, $X_0 \subset X$ a Cartier divisor and $x \in X_0$ a closed point. Assume that $\dim_x X \geq 4$. What can one say about the kernel of the restriction map between the local Picard groups

$$\mathrm{rest}_{X_0}^X : \mathrm{Pic}^{\mathrm{loc}}(x \in X) \rightarrow \mathrm{Pic}^{\mathrm{loc}}(x \in X_0)? \quad (1.1)$$

We consider three conjectural answers to this question.

- (2) The map (1.1) is an injection if X_0 is S_2 .
- (3) The kernel is p -torsion if X is an excellent, local \mathbb{F}_p -algebra.
- (4) The kernel is contained in the connected subgroup of $\mathrm{Pic}^{\mathrm{loc}}(x \in X)$.

The main result of this paper gives a positive answer to the topological variant (1.4) in some cases. The precise conditions in Theorem 13 are technical and they might even seem unrealistically special. Instead of stating them, I focus on three applications first.

My interest in this subject started with trying to understand higher dimensional analogs of three theorems and examples concerning surface singularities and their deformations; see [Ses75, Art76, Har10] for introductions. The main results of this note imply that none of them occurs for isolated singularities in dimensions ≥ 3 .

2 (Three phenomena in the deformation theory of surfaces).

- (1) There is a projective surface S_0 with quotient singularities and ample canonical class such that S_0 has a smoothing $\{S_t : t \in \mathbb{D}\}$ where S_t is a rational surface for $t \neq 0$.

Explicit examples were written down only recently (see [LP07, PPS09a, PPS09b] or the simpler [Kol13, 3.76]), but it has been known for a long time that K^2 can jump in flat families of surfaces with quotient singularities.

The simplest such example is classical and was known to Bertini (though he probably did not consider K^2 for a singular surface). Let $C_4 \subset \mathbb{P}^5$ be the cone over the degree 4 rational normal curve in \mathbb{P}^4 . It has two different smoothings. In one family $\{S_t : t \in \mathbb{D}\}$ (where $S_0 := C_4$) the general fiber is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)$. In the other family $\{R_t : t \in \mathbb{D}\}$ (where $R_0 := C_4$) the general fiber is $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by $\mathcal{O}_{\mathbb{P}^2}(2)$.

Note that $K_{R_t}^2 = 9$ and $K_{S_t}^2 = 8$ for $t \neq 0$, thus K^2 jumps in one of the families. In fact, it is easy to compute that $K_{C_4}^2 = 9$, so the jump occurs in the family $\{S_t : t \in \mathbb{D}\}$.

- (2) There are non-normal, isolated, smoothable surface singularities ($0 \in S_0$) whose normalization is simple elliptic [Mum78].
- (3) Every rational surface singularity ($0 \in S_0$) has a smoothing that admits a simultaneous resolution.

It is known that such smoothings form a whole component of the deformation space $\text{Def}(0 \in S_0)$, called the *Artin component* [Art74]. Generalizations of this can be used to describe all components of the deformation space of quotient singularities [KSB88], and, conjecturally, of all rational surface singularities and many other non-rational singularities [Kol91, dJvS92].

The higher dimensional versions of these were studied with the ultimate aim of compactifying the moduli space of varieties of general type; see [Kol12] for an introduction. The general theory of [KSB88, Kol12] suggests that one should work with *log canonical* singularities; see [KM98] or (9) for their definition and basic properties. This guides our generalizations of (2.1–2).

In order to develop (2.3) further, recall that a surface singularity is rational iff its divisor class group is finite; see [Mum61].

In this paper we state the results for normal varieties. In the applications to moduli questions one needs these results for semi-log canonical pairs (X, Δ) . All the theorems extend to this general setting using the methods of [Kol13, Chap.5]; see the forthcoming [Kol14, Chap.3] for details.

We say that a variety W is *smooth* (resp. *normal*) in codimension r if there is a closed subscheme $Z \subset W$ of codimension $\geq r+1$ such that $W \setminus Z$ is smooth (resp. normal).

Theorem 3. *None of the above examples (2.1–3) exists for varieties with isolated singularities in dimension ≥ 3 . More generally the following hold.*

- (1) *Let X_0 be a projective variety with log canonical singularities and ample canonical class. If X_0 is smooth in codimension 2 then every deformation of X_0 also has ample canonical class.*
- (2) *Let X_0 be a non-normal variety whose normalization is log canonical. If X_0 is normal in codimension 2 then X_0 is not smoothable, it does not even have normal deformations.*
- (3) *Let X_0 be a normal variety whose local class groups are torsion and $\{X_t : t \in \mathbb{D}\}$ a smoothing. If X_0 is smooth in codimension 2 then $\{X_t : t \in \mathbb{D}\}$ does not admit a simultaneous resolution.*

Our results in Sections 3–4 are even stronger; we need only some control over the singularities in codimension 2.

Such results have been known if X_0 (or its normalization) has rational singularities. These essentially follow from [Gro68, XI.3.16]; see [Kol95] for details. Thus the new part of Theorem 3.1–2 is that the claims also hold for log canonical singularities that are not rational.

Further comments and problems.¹

The theorems of SGA rarely have unnecessary assumptions, so an explanation is needed why Problem 1 could be an exception. One reason is that while our assumptions are weaker, the conclusions in [Gro68] are stronger.

Theorem 4. [Gro68, XI.2.2] *Let X be a scheme of pure dimension $n+1$, $X_0 \subset X$ a Cartier divisor and $Z \subset X$ a closed subscheme such that $Z_0 := X_0 \cap Z$ has dimension $\leq n-3$. Let D^* be a Cartier divisor on $X \setminus Z$ such that $D^*|_{X_0 \setminus Z_0}$ extends to a Cartier divisor on X_0 . Assume furthermore that X_0 is S_2 and $\text{depth}_{Z_0} \mathcal{O}_{X_0} \geq 3$. Then D^* extends to a Cartier divisor on X , in some neighborhood of X_0 . \square*

In Problem 1 we assume that Z is contained in X_0 , thus it is not entirely surprising that the depth condition $\text{depth}_{Z_0} X_0 \geq 3$, could be relaxed.

Example 12 shows that in Theorem 4 the condition $\text{depth}_{Z_0} X_0 \geq 3$ is necessary.

Another reason why Problem 1 may have escaped attention is that the topological version of it fails. In (28–29) we construct normal, projective varieties Y (of arbitrary large dimension) with a single singular point $y \in Y$ and a smooth hyperplane section H (not passing through y) such that the restriction map $H^2(Y, \mathbb{Q}) \rightarrow H^2(H, \mathbb{Q})$ is not injective. However, the kernel does not contain $(1, 1)$ -classes. In the example Y even has a log canonical singularity at y .

The arguments in Section 5 show that, at least over \mathbb{C} , a solution to Problem 1 would be implied by the following.

Problem 5. Let W be a normal Stein space of dimension ≥ 3 and L a holomorphic line bundle on W . Assume that there is a compact set $K \subset W$ such that the restriction of $c_1(L)$ is zero in $H^2(W \setminus K, \mathbb{Z})$.

Does this imply that $c_1(L)$ is zero in $H^2(W, \mathbb{Z})$?

Another approach would be to use intersection cohomology to restore Poincaré duality in (26.2). For this to work, the solution of the following is needed.

Problem 6. Let W be a normal analytic space. Is there an exact sequence

$$H^1(W, \mathcal{O}_W) \rightarrow \text{Pic}^{\text{an}}(W) \otimes \mathbb{Q} \rightarrow IH^2(W, \mathbb{Q})? \quad (6.1)$$

Note first that the sequence exists. Indeed, let $g : W' \rightarrow W$ be a resolution of singularities. If L is a line bundle on W , then g^*L is a line bundle on W' hence it has a Chern class $c_1(g^*L) \in H^2(W', \mathbb{Z})$. By the decomposition theorem [BB82] $IH^2(W, \mathbb{Q})$ is a direct summand of $H^2(W', \mathbb{Q})$ (at least for algebraic varieties).

Arapura explained to me that the sequence (6.1) should be exact for projective varieties by weight considerations but the general complex case is not clear. For our applications we need the case when W is Stein.

¹Recently Bhattacharya and de Jong proved Conjecture 1.3 in general and Conjecture 1.2 for schemes essentially of finite type in characteristic 0.

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1. DEFINITIONS AND EXAMPLES

Definition 7 (Local Picard groups). Let X be a scheme and $p \in X$ a point. The *local Picard group* $\mathrm{Pic}^{\mathrm{loc}}(p \in X)$ is a group whose elements are S_2 sheaves F on some neighborhood $p \in U \subset X$ such that F is locally free on $U \setminus \{p\}$. Two such sheaves give the same element if they are isomorphic over some neighborhood of p . The product is given by the S_2 -hull of the tensor product.

One can also realize the local Picard group as the direct limit of $\mathrm{Pic}(U \setminus \{p\})$ as U runs through all open Zariski neighborhoods of p or as $\mathrm{Pic}(\mathrm{Spec} \mathcal{O}_{x,X} \setminus \{p\})$.

If X is normal and $X \setminus \{x\}$ is smooth then $\mathrm{Pic}^{\mathrm{loc}}(p \in X)$ is isomorphic to the divisor class group of $\mathcal{O}_{x,X}$.

In many contexts it is more natural to work with the *étale-local Picard group* $\mathrm{Pic}^{\mathrm{et-loc}}(p \in X) := \mathrm{Pic}(\mathrm{Spec} \mathcal{O}_{x,X}^h \setminus \{p\})$ where $\mathcal{O}_{x,X}^h$ is the Henselization of the local ring $\mathcal{O}_{x,X}$.

If X is defined over \mathbb{C} , let $W \subset X$ be the intersection of X with a small (open) ball around p . The *analytic local Picard group* $\mathrm{Pic}^{\mathrm{an-loc}}(p \in X)$ can be defined as above using (analytic) S_2 sheaves on W . By [Art69], there is a natural isomorphism

$$\mathrm{Pic}^{\mathrm{et-loc}}(p \in X) \cong \mathrm{Pic}^{\mathrm{an-loc}}(p \in X).$$

Note that $\mathrm{Pic}^{\mathrm{an}}(W \setminus \{p\})$ is usually much bigger than $\mathrm{Pic}^{\mathrm{an-loc}}(p \in W)$. (This happens already for $X = \mathbb{C}^2$.) However, $\mathrm{Pic}^{\mathrm{an}}(W \setminus \{p\}) = \mathrm{Pic}^{\mathrm{an-loc}}(p \in X)$ if $\mathrm{depth}_p \mathcal{O}_X \geq 3$ [Siu69].

(The literature does not seem to be consistent; any of the above four variants is called the local Picard group by some authors.)

Let X be a complex space and $p \in X$ a closed point. Set $U := X \setminus \{p\}$. As usual, $\mathrm{Pic}(U) \cong H^1(U, \mathcal{O}_U^*)$ and the exponential sequence

$$0 \rightarrow \mathbb{Z}_U \xrightarrow{2\pi i} \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \rightarrow 1$$

gives an exact sequence

$$H^1(U, \mathcal{O}_U) \rightarrow \mathrm{Pic}(U) \xrightarrow{c_1} H^2(U, \mathbb{Z}).$$

A piece of the local cohomology exact sequence is

$$H^1(X, \mathcal{O}_X) \rightarrow H^1(U, \mathcal{O}_U) \rightarrow H_p^2(X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X).$$

Thus if X is Stein then we have an isomorphism

$$H^1(U, \mathcal{O}_U) \cong H_p^2(X, \mathcal{O}_X)$$

and the latter vanishes iff $\mathrm{depth}_p \mathcal{O}_X \geq 3$; see [Gro67, Sec.3]. Combining with [Siu69] we obtain the following well known result.

Lemma 8. *Let X be a Stein space and $p \in X$ a closed point. If $\mathrm{depth}_p \mathcal{O}_X \geq 3$ then taking the first Chern class gives an injection*

$$c_1 : \mathrm{Pic}^{\mathrm{an-loc}}(p \in X) \hookrightarrow H^2(X \setminus \{p\}, \mathbb{Z}). \quad \square$$

Definition 9 (Log canonical singularities). (See [KM98] for an introduction and [Kol13] for a comprehensive treatment of these singularities.)

Let X be a normal variety such that mK_X is Cartier for some $m > 0$. Let $f : Y \rightarrow X$ be a resolution of singularities with exceptional divisors $\{E_i : i \in I\}$. One can then write

$$mK_Y \sim f^*(mK_X) + m \cdot \sum_{i \in I} a(E_i, X) E_i.$$

The number $a(E_i, X) \in \mathbb{Q}$ is called the *discrepancy* of E_i ; it is independent of the choice of m .

If $\min\{a(E_i, X) : i \in I\} \geq -1$ then the minimum is independent of the resolution $f : Y \rightarrow X$ and its value is called the *discrepancy* of X .

X is called *log canonical* if $\min\{a(E_i, X) : i \in I\} \geq -1$ and *log terminal* if $\min\{a(E_i, X) : i \in I\} > -1$. A cone over a smooth variety with trivial canonical class is log canonical but not log terminal.

Let X be log canonical, $g : X' \rightarrow X$ any resolution and $E \subset X'$ an exceptional divisor such that $a(E, X) = -1$. The subvariety $g(E) \subset X$ is called a *log canonical center* of X . Log canonical centers hold the key to understanding log canonical varieties, see [Kol13, Chaps.4–5].

Log terminal singularities are rational [KM98, 5.22]. Log canonical singularities are usually not rational but they are Du Bois [KK10].

Log canonical singularities (and their non-normal versions, called semi-log canonical singularities) are precisely those that are needed to compactify the moduli of varieties of general type.

We use only the following two theorems about log canonical singularities.

Theorem 10. [Kaw07, Ale08] *Let X be a normal variety over a field of characteristic 0, $Z \subset X$ a closed subscheme of codimension ≥ 3 and $Z \subset X_0 \subset X$ a Cartier divisor such that $X_0 \setminus Z$ is normal and the normalization of X_0 is log canonical. Assume also that K_X is \mathbb{Q} -Cartier. Then X_0 is normal and it does not contain any log canonical center of X . \square*

Theorem 11. *Let X be a normal variety over a field of characteristic 0, $Z \subset X$ a closed subscheme of codimension ≥ 3 and D a Weil divisor on X that is Cartier on $X \setminus Z$.*

Then there is a proper, birational morphism $f : Y \rightarrow X$ such that f is small (that is, its exceptional set has codimension ≥ 2) and the birational transform $f_^{-1}D$ is \mathbb{Q} -Cartier and f -ample if one of the following assumptions is satisfied.*

- (1) [KSB88] *There is a Cartier divisor $Z \subset X_0 \subset X$ such that $X_0 \setminus Z$ is normal and the normalization of X_0 is log canonical.*
- (2) [Bir11, HX13, OX12] *X is log canonical and Z does not contain any log canonical center of X .* \square

Observe that D is \mathbb{Q} -Cartier iff $f : Y \rightarrow X$ is an isomorphism. In our applications we show that $Y \neq X$ leads to a contradiction.

Comments on the references. [Kaw07] proves that the pair (X, X_0) is log canonical. This implies that X_0 does not contain any log canonical center of X by an easy monotonicity argument [KM98, 2.27]. Then [Ale08] shows that X_0 is S_2 , hence normal since we assumed normality in codimension 1.

[KSB88] claimed (11.1) only for $\dim X = 3$ since the necessary results of Mori's program were known only for $\dim X \leq 3$ at that time. The proof of the general case is the same.

The second case (11.2) is not explicitly claimed in the references but it easily follows from them. For details on both cases see [Kol13, Sec.1.4].

The next example shows that Theorem 4 fails if $\operatorname{depth}_{Z_0} X_0 < 3$, even if the dimension is large.

Example 12. Let (A, Θ) be a principally polarized Abelian variety over a field k . The affine cone over A with vertex v is

$$C_a(A, \Theta) := \operatorname{Spec}_k \sum_{m \geq 0} H^0(A, \mathcal{O}_A(m\Theta)).$$

Note that $\operatorname{depth}_v C_a(A, \Theta) = 2$ since $H^1(A, \mathcal{O}_A) \neq 0$.

Set $X := C_a(A, \Theta) \times \operatorname{Pic}^0(A)$ with $f : X \rightarrow \operatorname{Pic}^0(A)$ the second projection. Since $L(\Theta)$ has a unique section for every $L \in \operatorname{Pic}^0(A)$, there is a unique divisor D_A on $A \times \operatorname{Pic}^0(A)$ whose restriction to $A \times \{[L]\}$ is the above divisor. By taking the cone we get a divisor D_X on X .

For $L \in \operatorname{Pic}^0(A)$, let $D_{[L]}$ denote the restriction of D_X to the fiber $C_a(A, \Theta) \times \{[L]\}$ of f . We see that

- (1) $D_{[L]}$ is Cartier iff $L \cong \mathcal{O}_A$.
- (2) $mD_{[L]}$ is Cartier iff $L^m \cong \mathcal{O}_A$.
- (3) $D_{[L]}$ is not \mathbb{Q} -Cartier for very general $L \in \operatorname{Pic}^0(A)$.

2. THE MAIN TECHNICAL THEOREM

The following is our main result concerning Problem 1. In the applications the key question will be the existence of the bimeromorphic morphism $f : Y \rightarrow X$. This is a very hard question in general, but in our cases existence is guaranteed by Theorem 11.

Theorem 13. *Let $f : Y \rightarrow X$ be a proper, bimeromorphic morphism of normal analytic spaces of dimension ≥ 4 and L a line bundle on Y whose restriction to every fiber is ample. Assume that there is a closed subvariety $Z_Y \subset Y$ of codimension ≥ 2 such that $Z := f(Z_Y)$ has dimension ≤ 1 and f induces an isomorphism $Y \setminus Z_Y \cong X \setminus Z$.*

Let $X_0 \subset X$ be a Cartier divisor such that $Z \cap X_0$ is a single point p . Let $p \in U \subset X$ be a contractible open neighborhood of p . Note that $U \setminus Z \cong f^{-1}(U) \setminus Z_Y$, hence the restriction $L|_{U \setminus Z}$ makes sense. Set $U_0 := X_0 \cap U$. The following are equivalent.

- (1) *The map f is an isomorphism over U .*
- (2) *The Chern class of $L|_{U \setminus Z}$ vanishes in $H^2(U \setminus Z, \mathbb{Q})$.*
- (3) *The Chern class of $L|_{U_0 \setminus \{p\}}$ vanishes in $H^2(U_0 \setminus \{p\}, \mathbb{Q})$.*

Proof. If f is an isomorphism then L is a line bundle on the contractible space U hence $c_1(L) = 0$ in $H^2(U, \mathbb{Q})$. Thus (2) holds and clearly (2) implies (3). The key part is to prove that (3) \Rightarrow (1).

The assumption and the conclusion are both local near p in the Euclidean topology. By shrinking X we may assume that the Cartier divisor X_0 gives a morphism $g : X \rightarrow \mathbb{D}$ to the unit disc \mathbb{D} whose central fiber is X_0 . Note that $Y_0 := f^{-1}(X_0)$ is a Cartier divisor in Y .

Let $W \subset X$ be the intersection of X with a closed ball of radius $0 < \epsilon \ll 1$ around p . Set $W_0 := X_0 \cap W$, $V := f^{-1}(W)$ and $V_0 := f^{-1}(W_0)$. We may assume that W, W_0 are contractible and $f^{-1}(p)$ is a strong deformation retract of both V and of V_0 .

Let $\bar{\mathbb{D}}_\delta \subset \mathbb{D}$ denote the closed disc of radius δ . If $0 < \delta \ll \epsilon$ then the pair $(W_0, \partial W_0)$ is a strong deformation retract of $(W \cap g^{-1}\bar{\mathbb{D}}_\delta, \partial W \cap g^{-1}\bar{\mathbb{D}}_\delta)$ and $(V_0, \partial V_0)$ is a strong deformation retract of $(V \cap (gf)^{-1}\bar{\mathbb{D}}_\delta, \partial V \cap (gf)^{-1}\bar{\mathbb{D}}_\delta)$. These retractions induce continuous maps (unique up-to homotopy)

$$r_c : (V_c, \partial V_c) \rightarrow (V_0, \partial V_0), \quad (13.4)$$

where V_c is the fiber of $g \circ f : V \rightarrow \mathbb{D}$ over $c \in \bar{\mathbb{D}}_\delta$. The induced maps

$$r_c^* : \mathbb{Q} \cong H^{2n}(V_0, \partial V_0, \mathbb{Q}) \xrightarrow{\cong} H^{2n}(V_c, \partial V_c, \mathbb{Q}) \cong \mathbb{Q} \quad (13.5)$$

are isomorphisms where $n = \dim_{\mathbb{C}} V_0 = \dim_{\mathbb{C}} V_c$. Our aim is to study the cup product pairing

$$H^2(V_0, \partial V_0, \mathbb{Q}) \times H^{2n-2}(V_0, \mathbb{Q}) \rightarrow H^{2n}(V_0, \partial V_0, \mathbb{Q}) \cong \mathbb{Q}. \quad (13.6)$$

(See [Hat02], especially pages 209 and 240 for the relevant facts on cup and cap products.) We prove in Lemmas 14 and 15, by arguing on V_c , that it is zero and in Lemma 17.2, by arguing on V_0 , that it is nonzero if $V_0 \rightarrow W_0$ is not finite. Thus $f^{-1}(p)$ is 0-dimensional, hence f is a biholomorphism. \square

For later applications, in the next lemmas we consider the more general case when $f : Y \rightarrow X$ is a proper, bimeromorphic morphism of normal analytic spaces and $g : X \rightarrow \mathbb{D}$ a flat morphism of relative dimension n .

Lemma 14. *Notation and assumptions as in (13). If $H_{2n-2}(V_c, \mathbb{Q}) = 0$ then the cup product pairing*

$$H^2(V_0, \partial V_0, \mathbb{Q}) \times H^{2n-2}(V_0, \mathbb{Q}) \rightarrow H^{2n}(V_0, \partial V_0, \mathbb{Q}) \cong \mathbb{Q} \text{ is zero.}$$

Proof. Using r_c^* and the Poincaré duality map, the cup product pairing factors through the following cup and cap product pairings, where the right hand sides are isomorphic by (13.5),

$$\begin{array}{ccccc} H^2(V_0, \partial V_0, \mathbb{Q}) & \times & H^{2n-2}(V_0, \mathbb{Q}) & \rightarrow & H^{2n}(V_0, \partial V_0, \mathbb{Q}) \cong \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \parallel \\ H^2(V_c, \partial V_c, \mathbb{Q}) & \times & H^{2n-2}(V_c, \mathbb{Q}) & \rightarrow & H^{2n}(V_c, \partial V_c, \mathbb{Q}) \cong \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \parallel \\ H_{2n-2}(V_c, \mathbb{Q}) & \times & H^{2n-2}(V_c, \mathbb{Q}) & \rightarrow & H_0(V_c, \mathbb{Q}) \cong \mathbb{Q} \end{array}$$

The first factor in the bottom row is zero, hence the pairing is zero. \square

We apply the next result to $V_c \rightarrow W_c$ to check the homology vanishing assumption in Lemma 14.

Lemma 15. *Let $V' \rightarrow W'$ be a proper bimeromorphic map of normal complex spaces of dimension $n \geq 3$. Assume that every fiber has complex dimension $\leq n-2$ and W' is Stein. Then $H_{2n-2}(V', \mathbb{Q}) = 0$.*

Proof. Let $E' \subset V'$ denote the exceptional set and $F' \subset W'$ its image. Then $\dim F' \leq n-2$, hence the exact sequence

$$H_{2n-2}(F', \mathbb{Q}) \rightarrow H_{2n-2}(W', \mathbb{Q}) \rightarrow H_{2n-2}(W', F', \mathbb{Q}) \rightarrow H_{2n-3}(F', \mathbb{Q})$$

shows that $H_{2n-2}(W', \mathbb{Q}) \cong H_{2n-2}(W', F', \mathbb{Q})$. The latter group is in turn isomorphic to $H_{2n-2}(V', E', \mathbb{Q})$ which sits in an exact sequence

$$H_{2n-2}(E', \mathbb{Q}) \rightarrow H_{2n-2}(V', \mathbb{Q}) \rightarrow H_{2n-2}(V', E', \mathbb{Q}).$$

Here $H_{2n-2}(E', \mathbb{Q})$ is generated by the fundamental classes of the compact irreducible components of E' , but we assumed that there are no such. Thus we have an injection

$$H_{2n-2}(V', \mathbb{Q}) \hookrightarrow H_{2n-2}(W', \mathbb{Q}).$$

Since W' is Stein and $2n - 2 > n$, Theorem 16 implies that $H_{2n-2}(W', \mathbb{Q}) = 0$. Thus we conclude that $H_{2n-2}(V', \mathbb{Q}) = 0$. \square

During the proof we have used the following.

Theorem 16. [Ham83, Ham86] or [GM88, p.152]. *Let W be a Stein space of dimension n . Then $H_i(W, \mathbb{Z})$ and $H^i(W, \mathbb{Z})$ both vanish for $i > n$. More generally, W is homotopic to a CW complex of dimension $\leq n$.* \square

Next we describe two cases when the cup product pairing (13.6) is nonzero. The first of these is used in Proposition 22 and the second in Theorem 13.

Lemma 17. *Let $f_0 : V_0 \rightarrow W_0$ be a projective, bimeromorphic morphism between irreducible complex spaces. Let $p \in W_0$ be a point. Assume that f_0 is an isomorphism over $W_0 \setminus \{p\}$ and $\dim f_0^{-1}(p) > 0$. Assume furthermore that one of the following holds.*

- (1) *There is a nonzero \mathbb{Q} -Cartier divisor $E_0 \subset V_0$ supported on $f_0^{-1}(p)$.*
- (2) *There is an f_0 -ample line bundle L such that $c_1(L)|_{\partial V_0} = 0$.*

Then $f_0^{-1}(p)$ has codimension 1 in V_0 and the cup product pairing

$$H^2(V_0, \partial V_0, \mathbb{Q}) \times H^{2n-2}(V_0, \mathbb{Q}) \rightarrow H^{2n}(V_0, \partial V_0, \mathbb{Q}) \cong \mathbb{Q} \quad \text{is nonzero.}$$

Proof. Consider first Case (1). Then $E_0 \neq 0$ shows that $f_0^{-1}(p)$ has codimension 1.

Let H be a relatively very ample line bundle. We have $c_1(E_0) \in H^2(V_0, \partial V_0, \mathbb{Q})$ and $c_1(H) \in H^2(V_0, \mathbb{Q})$. If E_0 is effective then

$$c_1(E_0) \cup c_1(H)^{n-1} = c_1(H|_{E_0})^{n-1} \in H_0(E_0, \mathbb{Q}) \rightarrow H_0(V_0, \mathbb{Q}) \quad (17.3)$$

is positive. If E_0 is not assumed effective then we claim that

$$c_1(E_0) \cup c_1(E_0) \cup c_1(H)^{n-2} \in H^{2n}(V_0, \partial V_0, \mathbb{Q}) \quad \text{is nonzero.}$$

The complete intersection of $(n-2)$ general members of H gives an algebraic surface S , proper over W_0 such that $E_0 \cap S$ is a nonzero linear combination of exceptional curves. Thus, by the Hodge index theorem,

$$c_1(E_0)^2 \cup c_1(H)^{n-2} = c_1(E_0|_S)^2 < 0,$$

completing the proof of (1).

Next assume that (2) holds. By assumption we can lift $c_1(L)$ to $\tilde{c}_1(L) \in H^2(V_0, \partial V_0, \mathbb{Q})$. (The lifting is in fact unique, but this is not important for us.) From this we obtain a class

$$[\tilde{c}_1(L)] \in H_{2n-2}(V_0, \mathbb{Q}) = H_{2n-2}(f^{-1}(p), \mathbb{Q}) = \sum \mathbb{Q}[A_i] \quad (17.4)$$

where $A_i \subset f^{-1}(p)$ are the irreducible components of dimension $n - 1$. So far we have not established that $\dim f^{-1}(p) = n - 1$, thus the sum in (17.4) could be empty. The key step is the following.

Claim 17.5. $[\tilde{c}_1(L)] = \sum a_i[A_i]$ where $a_i < 0$ for every i and the sum is not empty.

Once this is shown we conclude as in (17.3) using the equality

$$\tilde{c}_1(L) \cup c_1(L)^{n-1} = \sum a_i \cdot c_1(L|_{A_i})^{n-1} < 0. \quad (17.6)$$

In order to prove (17.5) we aim to use [KM98, 3.39], except there it is assumed that every A_i is \mathbb{Q} -Cartier. To overcome this, take a resolution $\pi : V'_0 \rightarrow V_0$ and write the homology class $[\tilde{c}_1(\pi^*L)]$ is a linear combination $\sum a'_i[A'_i]$ where $A'_i \subset (\pi \circ \pi)^{-1}(p)$ are the irreducible components of dimension $n - 1$. Since L is f -ample and $\dim f^{-1}(p) > 0$, we see that π^*L is nef and not numerically trivial on $(\pi \circ \pi)^{-1}(p)$. Apply [KM98, 3.39.2] to π^*L . We obtain that $-\sum a'_i[A'_i]$ is effective and its support contains $(\pi \circ \pi)^{-1}(p)$. Thus $a'_i < 0$ for every i and so $[\tilde{c}_1(L)] = \pi_* \sum a'_i[A'_i]$ shows (17.5) unless there are no f -exceptional divisors $A_i \subset f^{-1}(p)$.

If this happens, then $\sum a'_i[A'_i]$ is π -exceptional and, as the homology class of π^*L , it has zero intersection with every curve that is contracted by π . Thus we can apply [KM98, 3.39.1] to both $\pm \sum a'_i[A'_i]$ and conclude that $\sum a'_i[A'_i] = 0$. This is a contradiction since L and hence π^*L have positive intersection with some curve. \square

3. DEFORMATIONS OF LOG CANONICAL SINGULARITIES

Here we derive stronger forms of the three assertions of Theorem 3. We start with (3.1–2).

Theorem 18. *Let X be a normal variety over \mathbb{C} and $g : X \rightarrow C$ a flat morphism of pure relative dimension n to a smooth curve. Let $0 \in C$ be a point and $Z_0 \subset X_0$ a closed subscheme of dimension $\leq n - 3$. Assume that*

- (1) K_X is \mathbb{Q} -Cartier on $X \setminus Z_0$,
- (2) the fibers X_c are log canonical for $c \neq 0$,
- (3) $X_0 \setminus Z_0$ is log canonical and
- (4) the normalization of X_0 is log canonical.

Then X_0 is normal and K_X is \mathbb{Q} -Cartier on X .

Proof. By localization we may assume that $Z_0 = \{p\}$ is closed point. Next we use Theorem 11 to obtain $f : Y \rightarrow X$ such that f is an isomorphism over $X \setminus \{p\}$ and $f_*^{-1}K_X$ is an f -ample \mathbb{Q} -Cartier divisor.

By the Lefschetz principle, we may assume that everything is defined over \mathbb{C} .

We apply Theorem 13 to $L := mf_*^{-1}K_X$ for a suitable $m > 0$. Let U be the intersection of X with a small ball around p and set $U_0 := X_0 \cap U$. Note that U_0 is naturally a subset of X , of Y and also of the normalization of X_0 . The latter shows that $L_{U_0 \setminus \{p\}}$ is trivial, thus the assumption (13.3) is satisfied. Hence f is an isomorphism and so K_X is \mathbb{Q} -Cartier.

Now Theorem 10 implies that X_0 is normal. \square

Theorem 19. *Let X be a log canonical variety of dimension ≥ 4 over \mathbb{C} and $p \in X$ a closed point that is not a log canonical center (9). Let $p \in X_0 \subset X$ be a Cartier*

divisor. Let $p \in U$ be a Stein neighborhood such that U and $U_0 := X_0 \cap U$ are both contractible. Then the restriction maps

$$\mathrm{Pic}^{\mathrm{loc}}(p \in X) \rightarrow \mathrm{Pic}^{\mathrm{loc}}(p \in X_0) \quad \text{and} \quad \mathrm{Pic}^{\mathrm{loc}}(p \in X) \rightarrow H^2(U_0 \setminus \{p\}, \mathbb{Z})$$

are injective.

Proof. Let D be a divisor on X such that $D|_{X \setminus \{p\}}$ is Cartier and $c_1(D|_{X_0 \setminus \{p\}})$ is zero in $H^2(X_0 \setminus \{p\}, \mathbb{Z})$.

First we show that D is \mathbb{Q} -Cartier at p . By Theorem 11 there is a proper birational morphism $f : Y \rightarrow X$ such that f is an isomorphism over $X \setminus \{p\}$, $f_*^{-1}D$ is f -ample and f has no exceptional divisors.

The Cartier divisor X_0 gives a morphism $X \rightarrow \mathbb{D}$ whose central fiber is X_0 .

As in Theorem 13 let $W_0 \subset X_0$ be the intersection of X_0 with a closed ball of radius ϵ around p and $V_0 := f^{-1}(W_0)$. Set $n := \dim X_0$. By Lemma 14 we see that the cup product pairing

$$H^2(V_0, \partial V_0, \mathbb{Q}) \times H^{2n-2}(V_0, \mathbb{Q}) \rightarrow H^{2n}(V_0, \partial V_0, \mathbb{Q}) \cong \mathbb{Q} \quad \text{is zero.}$$

On the other hand, by (17.2) it is nonzero unless $f : Y \rightarrow X$ is finite. Thus f is an isomorphism and D is \mathbb{Q} -Cartier at p .

Now we can use [Gro68, X.3.2] to show that D is Cartier at p . \square

Corollary 20. *Let $g : X \rightarrow C$ and $Z_0 \subset X_0$ be as in Theorem 18. Assume that the fibers X_c are all log canonical and K_X is \mathbb{Q} -Cartier. Let D^* be Cartier divisor on $X \setminus Z_0$ such that $D^*|_{X_0 \setminus Z_0}$ extends to a Cartier divisor on X_0 .*

Then D^ extends to a Cartier divisor on X .*

Proof. Choose Z to be the smallest closed subset such that D^* is Cartier on $X \setminus Z$. We need to show that $Z = \emptyset$. If not, let $p \in Z$ be a generic point. By localization we are reduced to the case when $Z = \{p\}$ is a closed point of X . Note that p is not a log canonical center of X by Theorem 10.

Thus (20) is a special case of Theorem 19. \square

21 (Proof of Theorem 3.1–2). Let $g : X \rightarrow C$ be a flat, proper morphism to a smooth curve whose fibers are normal and log canonical. Let $0 \in C$ be a closed point and $Z_0 \subset X_0$ a subscheme of codimension ≥ 3 such that K_X is \mathbb{Q} -Cartier on $X \setminus Z_0$. Then K_X is \mathbb{Q} -Cartier by Corollary 20 thus mK_X is Cartier for some $m > 0$. So $\mathcal{O}_X(mK_X)$ is a line bundle on X . For a flat family of line bundles, ampleness is an open condition, proving (3.1).

The second assertion (3.2) directly follows from Theorem 18. \square

4. STABILITY OF EXCEPTIONAL DIVISORS

We consider part 3 of Theorem 3. Let $g : X \rightarrow C$ be a flat morphism to a smooth curve. Let $0 \in C$ be a closed point such that X_0 is \mathbb{Q} -factorial. Let $Z_0 \subset X_0$ be a subscheme of codimension ≥ 3 and $f : Y \rightarrow X$ be a projective, birational morphism such that f is an isomorphism over $X \setminus Z_0$ and $f_0 : Y_0 \rightarrow X_0$ is birational but not an isomorphism.

Let $H_0 \subset Y_0$ be an ample divisor. Since X_0 is \mathbb{Q} -factorial, $m \cdot f_0(H_0)$ is Cartier for some $m > 0$. Thus

$$E_0 := f_0^*(m f_0(H_0)) - m H_0$$

is a nonzero, f_0 -exceptional Cartier divisor. We will show that this implies that $Y_t \rightarrow X_t$ is not an isomorphism, contrary to our assumptions.

More generally, let Y_0 be a complex analytic space and $E_0 \subset Y_0$ a proper, complex analytic subspace. We would like to prove that, under certain conditions, every deformation $\{Y_t : t \in \mathbb{D}\}$ induces a corresponding deformation $\{E_t \subset Y_t : t \in \mathbb{D}\}$.

If E_0 is a Cartier divisor, then by deformation theory (see, for instance, [Kol96, Sec.I.2] or [Har10, Sec.6]) the obstruction space is $H^1(E_0, \mathcal{O}_{Y_0}(E_0)|_{E_0})$. If E_0 is smooth and its normal bundle is negative, then by Kodaira's vanishing theorem the obstruction group is zero, hence every flat deformation of Y_0 induces a flat deformation of the pair $(E_0 \subset Y_0)$.

Here we address the more general case when there is a projective morphism $f_0 : Y_0 \rightarrow X'_0$ which contracts E_0 to a point. (This always holds if the normal bundle of E_0 is negative, at least for analytic or algebraic spaces, see [Art70].) We allow E_0 to be singular. By [MR71], any flat deformation $\{Y_t : t \in \mathbb{D}\}$ induces a corresponding deformation $\{f_t : Y_t \rightarrow X_t : t \in \mathbb{D}\}$ (with the slight caveat that X_0 need not be normal, but its normalization is X'_0). We can state our result in a more general form as follows.

Proposition 22. *Let $g : X \rightarrow \mathbb{D}$ be a flat morphism of pure relative dimension n . Let $f : Y \rightarrow X$ be a projective, bimeromorphic morphism such that $f_0 : Y_0 \rightarrow X_0$ is also bimeromorphic.*

Assume that there is a nonzero (but not necessarily effective) \mathbb{Q} -Cartier divisor $E_0 \subset Y_0$ such that $\dim f_0(\text{Supp } E_0) \leq n - 3$.

Then, for every $|t| \ll 1$ there is a nonzero exceptional divisor $E_t \subset \text{Ex}(f_t)$.

Proof. By taking general hyperplane sections of X we may assume that $f_0(\text{Supp } E_0)$ is a point $p \in X_0$.

We use the notation of Theorem 13. Lemma 17 shows that the cup product pairing

$$H^2(V_0, \partial V_0, \mathbb{Q}) \times H^{2n-2}(V_0, \mathbb{Q}) \rightarrow H^{2n}(V_0, \partial V_0, \mathbb{Q}) \cong \mathbb{Q}$$

is nonzero. On the other hand, if $\dim \text{Ex}(f_t) \leq n - 2$ for $t \neq 0$ then Lemma 15 applies and so Lemma 14 shows that the above cup product pairing is zero, a contradiction. \square

Remark 23. (1) Note first that we do not assert that $\{E_t : t \in \mathbb{D}\}$ is a flat family of divisors, nor do we claim that the E_t are \mathbb{Q} -Cartier. Most likely both of these hold under some natural hypotheses.

(2) The dimension restriction $\dim f_0(\text{Supp } E_0) \leq n - 3$ is indeed necessary. If Y_0 is a smooth surface and $E_0 \subset Y_0$ is a smooth rational curve then the analog of Proposition 22 holds only if E_0 is a (-1) -curve.

(3) The existence of a \mathbb{Q} -Cartier divisor $E_0 \subset Y_0$ seems an unusual assumption, but it is necessary, as shown by the following examples.

Let W be any smooth projective variety of dimension n and L a very ample line bundle on W . Let Y be the total space of the rank $r \geq 2$ bundle $L^{-1} + \cdots + L^{-1}$ with zero section $W \cong E \subset Y$. Let $f : Y \rightarrow X$ be the contraction of W to a point; that is, X is the spectrum of the symmetric algebra $H^0(W, \text{Sym}(L + \cdots + L))$.

For any general map $g : X \rightarrow \mathbb{A}^1$ the conclusion of Proposition 22 fails since we have $Y \rightarrow X$. The fiber over the origin is a hypersurface $Y_0 \subset Y$ that contains E and the codimension of E in Y_0 is $r - 1$.

If $r > n$ then a general Y_0 is smooth but for these $\dim E \leq \frac{1}{2} \dim Y_0$.

If $r \leq n$ then Y_0 is always singular. The most interesting case is when $r = n$. Then, for general choices, the only singularities of Y_0 are ordinary nodes along E . If $n = r = 2$ then E is a divisor in Y_0 but it is not \mathbb{Q} -Cartier at these nodes.

An interesting special case arises when $W = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $r = n + 1$. Then X_0 has a terminal singularity and $Y_0 \rightarrow X_0$ is crepant.

(4) The conclusion of Proposition 22 should hold if f is only proper, but the current proof uses projectivity in an essential way.

(5) An examination of the proof shows that Proposition 22 can be extended to higher codimension exceptional sets as follows. In view of the examples in (3), the assumptions seem to be optimal.

Proposition 24. *Let $g : X \rightarrow \mathbb{D}$ be a flat morphism of pure relative dimension n . Let $f : Y \rightarrow X$ be a projective, bimeromorphic morphism such that $f_0 : Y_0 \rightarrow X_0$ is also bimeromorphic.*

Assume that $\text{Ex}(f_0)$ is mapped to a point, $d := \dim \text{Ex}(f_0) > n/2$ and $\text{Ex}(f_0)$ supports an effective d -cycle that is the Poincaré dual of a cohomology class in $H^{2(n-d)}(Y_0, \partial Y_0, \mathbb{Q})$. (The latter always holds if Y_0 is smooth.)

Then $\dim \text{Ex}(f_t) = d$ for every t . \square

5. ANOTHER TOPOLOGICAL APPROACH

The aim of this section is to recall the usual topological approach to Problem 1, going back at least to [Bot59]. This works if $X \setminus X_0$ is smooth. Then we discuss a possible modification of the method that could lead to a proof over \mathbb{C} .

25 (Set-up). Let $X \subset \mathbb{C}^N$ be an analytic space of pure dimension n and $X_0 \subset X$ a Cartier divisor. Let $p \in X_0$ be a point, $W \subset X$ the intersection of X with a small closed ball around p and set $W_0 := W \cap X_0$. We assume that

- (1) the interior of W is Stein,
- (2) $W \setminus \{p\}$ is homeomorphic to $\partial W \times [0, 1]$,
- (3) $W_0 \setminus \{p\}$ is homeomorphic to $\partial W_0 \times [0, 1]$ and
- (4) $X \setminus X_0$ is smooth.

Proposition 26. *Notation and assumptions as above. Then the natural map*

$$H^i(W \setminus \{p\}, \mathbb{Z}) \rightarrow H^i(W_0 \setminus \{p\}, \mathbb{Z})$$

is an isomorphism for $i \leq n - 3$ and an injection for $i = n - 2$.

Proof. By assumption,

$$H^i(W \setminus \{p\}, \mathbb{Z}) \cong H^i(\partial W, \mathbb{Z}) \quad \text{and} \quad H^i(W_0 \setminus \{p\}, \mathbb{Z}) \cong H^i(\partial W_0, \mathbb{Z}).$$

We have an exact sequence

$$H^i(\partial W, \partial W_0, \mathbb{Z}) \rightarrow H^i(\partial W, \mathbb{Z}) \rightarrow H^i(\partial W_0, \mathbb{Z}) \rightarrow H^{i+1}(\partial W, \partial W_0, \mathbb{Z}) \quad (26.1)$$

Since $W \setminus W_0$ is smooth, Poincaré duality shows that

$$H^i(\partial W, \partial W_0, \mathbb{Z}) = H_{2n-1-i}(\partial W \setminus \partial W_0, \mathbb{Z}) \cong H_{2n-1-i}(W \setminus W_0, \mathbb{Z}). \quad (26.2)$$

By assumption the interior of W is n -dimensional and Stein, hence so is the interior of $W \setminus W_0$. Thus $H_{2n-1-i}(W \setminus W_0, \mathbb{Z}) = 0$ for $2n - 1 - i \geq n + 1$ by Theorem 16.

If $i \leq n - 3$ then both groups at the end of (26.1) are zero, giving the isomorphism $H^i(W \setminus \{p\}, \mathbb{Z}) \cong H^i(W_0 \setminus \{p\}, \mathbb{Z})$.

If $i = n - 2$ then only the group on the left vanishes, thus we get only an injection $H^{n-2}(W \setminus \{p\}, \mathbb{Z}) \hookrightarrow H^{n-2}(W_0 \setminus \{p\}, \mathbb{Z})$. \square

Corollary 27. *Notation and assumptions as above. Assume in addition that $\operatorname{depth}_p \mathcal{O}_{X_0} \geq 2$ and $\dim X_0 \geq 3$. Then the natural restriction*

$$\operatorname{Pic}^{an}(W \setminus \{p\}) \rightarrow \operatorname{Pic}^{an}(W_0 \setminus \{p\})$$

is an injection.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Pic}^{an}(W \setminus \{p\}) & \hookrightarrow & H^2(W \setminus \{p\}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \operatorname{Pic}^{an}(W_0 \setminus \{p\}) & \rightarrow & H^2(W_0 \setminus \{p\}, \mathbb{Z}) \end{array}$$

The top horizontal arrow is injective by Lemma 8 and the right hand vertical arrow is injective by Proposition 26, hence the left hand vertical arrow is also injective. \square

The next lemma shows that, even in the classical setting, that is when Y is projective and $Y_0 \subset Y$ is an ample divisor, the restriction map $H^2(Y, \mathbb{Q}) \rightarrow H^2(Y_0, \mathbb{Q})$ is not injective under some conditions. We then show in (29) that such examples do exist.

Lemma 28. *Let X be a smooth projective variety of dimension n and $Z \subset X$ a smooth divisor. Assume that $H^1(X, \mathbb{Q}) = 0$ and there is a morphism $g : X \rightarrow Y$ that contracts Z to a point $y \in Y$ and is an isomorphism otherwise. Let $Y_0 \subset Y$ be a smooth divisor (not passing through y). Then the kernel of the restriction map $H^2(Y, \mathbb{Q}) \rightarrow H^2(Y_0, \mathbb{Q})$ contains $H^1(Z, \mathbb{Q})$.*

Proof. The cohomology sequence of the pair (Y, y) shows that $H^2(Y, \mathbb{Q}) \cong H^2(Y, y, \mathbb{Q})$ and $H^2(Y, y, \mathbb{Q}) \cong H^2(X, Z, \mathbb{Q})$. The latter in turn sits in an exact sequence

$$H^1(X, \mathbb{Q}) \rightarrow H^1(Z, \mathbb{Q}) \rightarrow H^2(X, Z, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q}).$$

We assumed that $H^1(X, \mathbb{Q}) = 0$ hence there is an injection

$$H^1(Z, \mathbb{Q}) \hookrightarrow \ker[H^2(X, Z, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})].$$

Since $H^2(Y, \mathbb{Q}) \rightarrow H^2(Y_0, \mathbb{Q})$ factors as

$$H^2(Y, \mathbb{Q}) \cong H^2(X, Z, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q}) \rightarrow H^2(Y_0, \mathbb{Q})$$

we obtain an injection

$$H^1(Z, \mathbb{Q}) \hookrightarrow \ker[H^2(Y, \mathbb{Q}) \rightarrow H^2(Y_0, \mathbb{Q})]. \quad \square$$

We thus need to find examples as above where $H^1(Z, \mathbb{Q}) \neq 0$. In the next examples, Z is an Abelian variety.

Example 29. Let $\pi_0 : S \rightarrow \mathbb{P}^1$ be a simply connected elliptic surface. For instance, we can take S to be the blow-up of \mathbb{P}^2 at the 9 base points of a cubic pencil.

By composing π with suitable automorphisms of \mathbb{P}^1 we get n simply connected elliptic surfaces $\pi_i : S_i \rightarrow \mathbb{P}^1$ such that for every point $p \in \mathbb{P}^1$ at most one of the π_i has a singular fiber over p . Thus the fiber product

$$X_1 := S_1 \times_{\mathbb{P}^1} S_2 \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} S_n$$

is a smooth variety of dimension $n+1$. General fibers of the projection $\pi_1 : X_1 \rightarrow \mathbb{P}^1$ are Abelian varieties of dimension n and X_1 is simply connected.

Fix Abelian fibers $A_1, A_2 \subset X_1$. Let $H_1 \subset X_1$ be a very ample divisor such that $A_1 \cap H_1$ is smooth. Let $X := B_{A_1 \cap H_1} X_1 \rightarrow X_1$ denote the blow-up of $A_1 \cap H_1$. Let

$H \subset X$ denote the birational transform of H_1 and $A \subset X$ the birational transform of A_1 .

For $m \gg 1$, the linear system $|H+mA_2|$ is base point free. This gives a morphism $g : X \rightarrow Y$. Note that g contracts A to a point $y \in Y$ and $g : X \setminus A \rightarrow Y \setminus \{y\}$ is an isomorphism.

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